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Technical Notes

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Explicit Kutta Condition for Unsteady Two-Dimensional High-Order Potential Boundary Element Method

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Introduction

DIFFERENT approaches are applied for the analysis of unsteady potential aerodynamic flow around a lifting body to find the numerical solution of the problem.[1–7] The potential-based direct approach relates the unknown potential on the surface of the body to the known boundary conditions. Usually, numerical solutions are obtained by using constant elements, and the discontinuity in potential at the trailing edge is approximated by the difference between the potentials evaluated at the centroidal control points of the contiguous elements at the trailing edge.[8,9] By doing so, the pressure continuity condition is violated at the trailing edge.[10] To account for this, an explicit Kutta condition should be applied at least for unsteady large-amplitude motion of low reduced frequency where the angle of attack is such that flow separation does not occur.[10,11] A method based on a Newton–Raphson scheme has been used by Kinnas and Hsin[10] to iterate an explicit equal pressure condition at the trailing edge. The implementation of this technique shows a slow convergence, and more recently a development based on the linearization of the pressure coefficient was employed by Bose.[11] Here, an explicit unsteady pressure Kutta condition is described that was directly and efficiently implemented in a time domain high-order potential panel method[12,13] so as to ensure the pressure equality on the upper and lower surfaces at the trailing edge of the airfoil at each time step.

Integral Equations and Unsteady Kutta Condition

The integral equation governing the problem at the time t is

$$c\phi = \int_{\partial\Omega_b}\left(\phi\frac{\partial G}{\partial n} - G\frac{\partial\phi}{\partial n}\right)\mathrm{d}s + \int_{\partial\Omega_w}\Delta\phi\frac{\partial G}{\partial n}\,\mathrm{d}s \tag{1}$$

where ϕ is the velocity potential; $\partial\Omega_b$ and $\partial\Omega_w$ are the boundaries of the airfoil and the wake, respectively; $\Delta\phi$ is the jump in potential; G is the singular solution of the unlimited domain; and c is a coefficient given by

$$c = \int_{\Omega_\infty}\nabla^2 G\,\mathrm{d}\Omega + \int_{\partial\Omega_b}\frac{\partial G}{\partial n}\,\mathrm{d}s \tag{2}$$

In the constant-potential panel methods for two-dimensional airfoils it is common to implement the Kutta condition by setting the jump in potential on the wake equal to

$$\Delta\phi = \phi_N - \phi_1 \tag{3}$$

where ϕ_N and ϕ_1 are the values of the potential at the centroids of the elements in contact with the trailing edge. Equation (3) does not ensure the pressure equality at the trailing edge because the pressure is a function of the velocity squared and the time rate of change of potential.[10,11] Note, however, that if the unknowns are assumed to be the values $\phi_{u,\mathrm{TE}}$ and $\phi_{l,\mathrm{TE}}$ of the potential ϕ at the upper and lower trailing edge, the jump in potential is exactly given by

$$\Delta\phi = \phi_{u,\mathrm{TE}} - \phi_{l,\mathrm{TE}} = \phi_{N+1} - \phi_1 \tag{4}$$

On the other hand, Eq. (1) at the trailing edge becomes, via a suitable limiting procedure,[12,13]

$$c_{\mathrm{I}}\phi_{l,\mathrm{TE}} + c_{\mathrm{II}}\phi_{u,\mathrm{TE}} = \int_{\partial\Omega_b}\left(\phi\frac{\partial G}{\partial n} - G\frac{\partial\phi}{\partial n}\right)\mathrm{d}s + \int_{\partial\Omega_w}\Delta\phi\frac{\partial G}{\partial n}\,\mathrm{d}s \tag{5}$$

where c_{I} and c_{II} are the coefficients obtained once the direction of the wake at the trailing edge is known. Because at the trailing edge only one equation is available for two unknowns, an explicit Kutta condition is required that is given by the pressure equality at the trailing edge,

$$\left[\frac{\partial\Delta\phi}{\partial t} + \frac{1}{2}\left(v_u^2 - v_l^2\right)\right]_{\mathrm{TE}} = 0 \tag{6}$$

A time-marching n-order boundary element scheme was numerically developed that allows us to implement exactly the explicit Kutta condition, i.e., Eq. (6), with respect to the nonlinear dependence of the pressure on the potential ϕ. The integral equation (1) can be expressed in numerical form, and a linear system of N algebraic equations is obtained. The N resulting equations involve the $N+1$ unknown potentials $\phi_1, \phi_2, \ldots, \phi_N, \phi_{N+1}$ at the airfoil nodes. The value of the airfoil circulation Γ at the time t is given by the difference between the potentials at the trailing edge, and then, from the trailing edge condition, $\Delta\phi_{\mathrm{TE}}$ on the wake is equal to Γ around the body. The resulting system is reordered to lead to

$$[\boldsymbol{A}][\phi] + [\boldsymbol{B}]\Gamma = [\boldsymbol{S}] + [\boldsymbol{Q}][\Delta\phi] \tag{7}$$

In Eq. (7) the vector $[\phi]$ contains the N unknown values of the potential on the body $\phi_1, \phi_2, \ldots, \phi_N$, the vector $[\Delta\phi]$ collects the values of the circulation at the previous time steps $\Delta\phi_1, \Delta\phi_2, \ldots, \Delta\phi_M$, whereas the vector $[\boldsymbol{S}]$ contains the contribution of the prescribed boundary data. The matrices $[\boldsymbol{A}]$, $[\boldsymbol{B}]$, and $[\boldsymbol{Q}]$ are the influence matrices. By inverting Eq. (7), one obtains

$$[\phi] = [\boldsymbol{A}]^{-1}([\boldsymbol{S}] + [\boldsymbol{Q}][\Delta\phi]) - [\boldsymbol{C}]\Gamma \tag{8}$$

where $[\boldsymbol{C}] = [\boldsymbol{A}]^{-1}[\boldsymbol{B}]$. Equation (8) expresses the N unknown potentials ϕ_i in terms of the circulation Γ around the airfoil at the time t, and then the potential and the velocity at any point can be expressed in terms of the unknown Γ. In particular, the perturbation velocities u on the airfoil are calculated through the derivatives of the potential expressed in terms of its nodal values through suitable shape functions H,

$$u(P) = [\boldsymbol{H}(\boldsymbol{P})][\phi] + H_\Gamma(P)\Gamma \qquad P \in \partial\Omega_b \tag{9}$$

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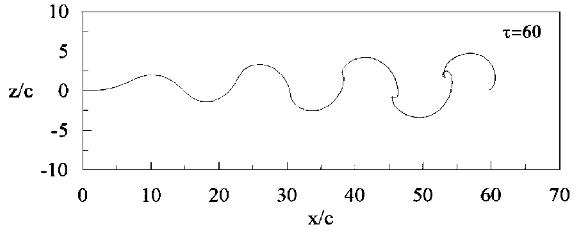

Fig. 1 Wake configuration for the NACA0012 foil undergoing a slow-frequency oscillation.

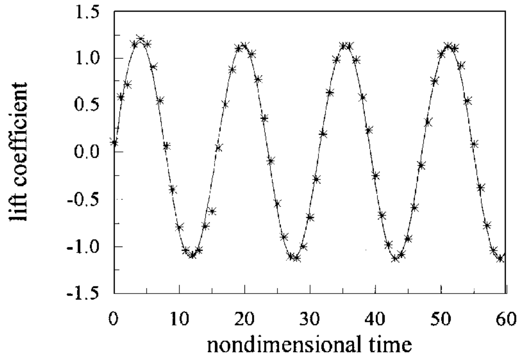

Fig. 2 Lift coefficient evolution for the NACA0012 foil at a reduced frequency of 0.2, with a feathering parameter of 0.4; ——, present boundary element method solution ($\Delta\tau = 0.5$) and *solution of Ref. 11.

Substituting Eq. (8) into Eq. (9), one has

$$u(P) = [\boldsymbol{H}(\boldsymbol{P})]\{[\boldsymbol{A}]^{-1}([\boldsymbol{S}] + [\boldsymbol{Q}][\Delta\phi]) - [\boldsymbol{C}]\Gamma\} + H_\Gamma(P)\Gamma \qquad P \in \partial\Omega_b \tag{10}$$

Therefore, the velocity can be expressed in a more compact form, with obvious meaning of the symbols, as

$$v(P) = w(P) + f(P)\Gamma \qquad P \in \partial\Omega_b \tag{11}$$

where the term $w(P)$ also contains the contribution due to the undisturbed flow. The amount of the circulation Γ is established by applying the unsteady Kutta condition [Eq. (6)], i.e., pressure equality at the trailing edge. In the discretized time domain, the $\partial\Delta\phi/\partial t$ term is numerically computed by the finite element scheme with respect to time after the potential discontinuity at the trailing edge was expressed through suitable shape functions $\Psi(t)$:

$$\Delta\phi_{\mathrm{TE}}(t) = [\boldsymbol{\Psi}(\boldsymbol{t})][\Delta\phi] + \Psi_\Gamma(t)\Gamma \tag{12}$$

Taking into account Eqs. (11) and (12), Eq. (6) becomes

$$\left(f_u^2 - f_l^2\right)_{\mathrm{TE}}\Gamma^2 + 2\left\{(f_u w_u - f_l w_l)_{\mathrm{TE}} + \dot{\Psi}_\Gamma\right\}\Gamma + \left\{\left(w_u^2 - w_l^2\right)_{\mathrm{TE}} + 2[\dot{\boldsymbol{\Psi}}][\Delta\phi]\right\} = 0 \tag{13}$$

where the overdot indicates the time derivative. Equation (13) is the desired equation to calculate the circulation around the lifting airfoil at each time step. It is the explicit pressure Kutta condition that ensures the equality of pressure at the trailing edge. Equation (13) is a quadratic algebraic equation; evidently only one of the two roots represents the natural value of the circulation Γ. The wake geometry is updated step by step through the velocity of the wake points and the vorticity $\Delta\phi$ convected according to the vorticity transport equation. The whole process may be repeated to obtain the circulation at the next time step. At $t = 0$, the wake was assumed to leave the airfoil along a prescribed direction. The solution scheme has been tested for the flowfield around a NACA0012 foil undergoing a sinusoidal oscillation with a heave amplitude ratio $h/c = 1$ and reduced frequency $\varpi = \omega c/2V_\infty$ of 0.2 (ω is the oscillation frequency and c is the chord length); the feathering parameter $\theta = V_\infty\alpha/\omega h$ (where α is the pitch amplitude) is assumed equal to 0.4. Calculations have been done discretizing the airfoil by 34 linear elements and using linear shape functions to interpolate the unknown boundary data. To investigate the present explicit Kutta condition, the pressures have been also calculated near the trailing edge from the velocity obtained through elements of second order. The results indicate that the difference between the two discretizations used is negligible. The development of the wake with its form and location at nondimensional time $\tau = tV_\infty/c$ equal to 60 is shown in Fig. 1. In Fig. 2 the variation in lift coefficient vs nondimensional time is shown, whereas in Fig. 3 the pressure coefficient distribution over the section for the considered motion is displayed at $\tau = 20$. The results obtained by the present approach are nearly coincident with those reported by Bose.[11] The present numerical method is capable of inherently ensuring equality of pressures at the trailing edge, and it is shown to be robust with respect to the size of the time step and the mesh refinement.

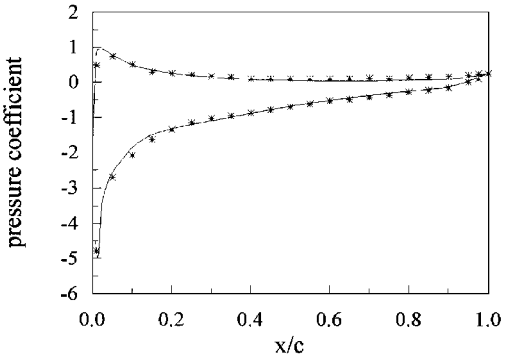

Fig. 3 Pressure coefficient distribution for the NACA0012 section at nondimensional time of 20; ——, present boundary element method solution and *solution of Ref. 11.

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